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NEAR-RESONANT MOTIONS IN SYSTEMS WITH RANDOM PERTURBATIONS†

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The effect of random perturbations on near-resonant motions in non-linear oscillatory systems is investigated. It is assumed that the equations of motion of the system can be reduced to standard form with a small parameter ε, and that an isolated primary resonance exists in the unperturbed system [1]. The behaviour of the perturbed system in the ε-neighbourhood of the resonance surface is considered and an effect analogous to deterministic "capture in resonance" [1] in an asymptotically long time interval is investigated.

Some special cases of resonances in diffusion systems in a plane were investigated previously in [2, 3]. The convergence of the perturbed motion to deterministic motion was proved and small deviations of the perturbed trajectories from a stationary point were investigated.

Below we show that random perturbations, that are small in the non-resonance domain, have a considerable effect on the behaviour of the trajectories close to resonance. Hence, in the near-resonance domain it is best to consider the trajectory as a whole as a random process rather than small random deviations from the unperturbed trajectory. A similar approach was developed for diffusion systems in [4] on the assumption that the width of the resonance domain is proportional to ε, rather than $\sqrt{\varepsilon}$. In this domain the capture in resonance effect cannot be observed.

In Section 1 we consider the basic model of a two-frequency system with random perturbations of an arbitrary kind. A successive averaging procedure is constructed which enables the slow variable to be separated. It is proved that in the near-resonance domain the equations of the perturbed motion can be reduced to the equation of an equivalent pendulum with a random torque. The zones of librations and rotations corresponding to passage through resonance and motion without intersecting the resonance surface are outlined, and a stochastic analogue of the "capture in resonance" effect is formulated. In Section 2 we formulate the necessary condition for "capture in resonance". In Section 3 we derive similar results for a multifrequency system. In Section 4 the theoretical results are used to analyse near-resonant motions in a system with one degree of freedom.

1. SEQUENTIAL AVERAGING

We will investigate a system of the form

$$\dot{x} = \varepsilon f(x, \theta_1, \theta_2) + \varepsilon F(x, \theta_1, \theta_2)\xi(t) \tag{1.1}$$

$$\dot{\theta}_i = \omega_i(x) + \varepsilon g_i(x, \theta_1, \theta_2) + \varepsilon G_i(x, \theta_1, \theta_2)\xi(t), \quad i = 1, 2$$

as $\varepsilon \to 0$. Here x and θ_i are scalars and $\xi(t)$ is a stationary (zero-mean) random process, satisfying certain mixing conditions [5], which are valid, in particular, for Gaussian processes. We will assume that the terms on the right-hand side of (1.1) can be represented as trigonometric polynomials, 2π-periodic in θ_i, with a finite number of harmonics and fairly smooth coefficients, and frequencies $\omega_i(x) \geqslant \bar{\omega}_i > 0$ over the domain considered. Non-resonant motions of (1.1) were investigated in [5].

We will compare (1.1) with the unperturbed system

$$\dot{x} = \varepsilon f(x, \theta_1, \theta_2), \quad \dot{\theta}_i = \omega_i(x) + \varepsilon g_i(x, \theta_1, \theta_2), \quad i = 1, 2 \tag{1.2}$$

and assume that in the approximation with respect to ε considered, system (1.2) has a single resonance surface

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$$\gamma(x)=\lambda_1\omega_1(x)+\lambda_2\omega_2(x)=0 \tag{1.3}$$

where λ_1 and λ_2 are certain integers, not simultaneously equal to zero. This means [6] that the time average

$$\langle f\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x,\omega_i(x)t+\varphi_i)dt,\quad i=1,2 \tag{1.4}$$

considered as a function of $\omega_i(x)$, has a discontinuity on surface (1.3).

As in deterministic systems, we will investigate the behaviour of solutions within the ε-neighbourhood of resonance surface (1.3). Suppose the initial point $x(0, \varepsilon) = \bar{x}$ lies in this neighbourhood, i.e.

$$\gamma(\bar{x})=\mu\bar{p},\quad \mu=\sqrt{\varepsilon} \tag{1.5}$$

Following [6], we will introduce the new variables

$$\gamma(x)=\lambda_1\omega_1(x)+\lambda_2\omega_2(x),\quad \Phi=\lambda_1\theta_1+\lambda_2\theta_2,\quad \theta_1=\psi \tag{1.6}$$

i.e. $x = X(\gamma)$. Seeking a solution which lies in the $\sqrt{\varepsilon}$-neighbourhood of (1.3), we put

$$\gamma(x)=\mu P,\quad \mu=\sqrt{\varepsilon} \tag{1.7}$$

Then (1.1) becomes

$$\begin{aligned}
\dot{P}&=\mu b(\mu P,\Phi,\psi)+\mu D(\mu P,\Phi,\psi)\xi(t)\\
\dot{\Phi}&=\mu P+\mu^2k(\mu P,\Phi,\psi)+\mu^2K(\mu P,\Phi,\psi)\xi(t)\\
\dot{\psi}&=\tilde{\omega}(\mu P)+\mu^2\upsilon(\mu P,\Phi,\psi)+\mu^2V(\mu P,\Phi,\psi)\xi(t)
\end{aligned} \tag{1.8}$$

and the coefficients (1.8) are obtained by substituting (1.6) and (1.7) into (1.1). Here $\tilde{\omega} = \omega_1(X(\mu P)) = \Omega_0 + \mu\Omega_1 P + \dots$, where $\Omega_0 = \omega_1(X(0)) = \text{const}$. Consequently, (1.8) can be regarded as a system in standard form, and we can apply to it the stochastic averaging method, taking higher approximations into account [7]. In accordance with [7], we must retain terms up to the second order in the deterministic coefficients, and terms of first-order infinitesimals in μ in the random coefficients. We will rewrite (1.8), taking into account only those terms that are important for the subsequent analysis

$$\begin{aligned}
&\dot{P}=\mu b_0(\Phi,\psi)+\mu^2b_1(\Phi,\psi)P+\mu D_0(\Phi,\psi)\xi(t)\\
&\dot{\Phi}=\mu P+\mu^2k_0(\Phi,\psi),\quad \dot{\psi}=\Omega_0+\mu\Omega_1P+\mu^2\upsilon_0(\Phi,\psi)
\end{aligned} \tag{1.9}$$

where $b_0(\Phi, \psi) = b(0, \Phi, \psi)$, $b_1(\Phi, \psi) = db/d(\mu P)|_{P=0}$, etc.

To analyse (1.9), we will construct an asymptotic procedure for separating motions which generalizes the successive averaging method developed for deterministic systems [8], and we will separate the equations for the slow variable which do not depend on Φ and ψ.

Averaging over the fast variable. Following [7], we will average (1.9) over ψ. We obtain that as $\mu \to 0$, the process $P(t, \mu)$ converges weakly [9] to the solution $p(t, \mu)$ of the averaged stochastic system

$$\dot{p}=\mu\beta_0(\varphi)+\mu^2\beta_1(\varphi)p+\mu\sigma_0(\varphi)\dot{w}(t),\quad \dot{\varphi}=\mu p+\mu^2\varkappa_0(\varphi) \tag{1.10}$$

with the same initial conditions as in (1.9). Here $w(t)$ is a standard Wiener process, where the drift and diffusion coefficients are given by the formulae [7]

$$\beta_i(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}b_i(\varphi,\psi)d\psi,\quad i=0,1,\quad \varkappa_0(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}k_0(\varphi,\psi)d\psi \tag{1.11}$$

$$\sigma_0^2(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\int_{-\infty}^{\infty}D_0(\varphi,\psi)D_0(\varphi,\psi+\Omega_0t)K_\xi(t)dt$$

and $K_\xi(t)$ is the correlation function of the process $\xi(t)$. The structure of the coefficients (1.9) are such that when averaging is carried out up to the second order, additional terms in the drift coefficient do not appear. It was proved in [7] that the convergence $P \to p$ is ensured in the time interval where the process $p(t, \mu)$ is weakly compact [9, 10]. Below we will show that weak compactness is ensured in the time interval $t \sim 1/\mu^2$.

Averaging over the semi-slow variable. In (1.10) we will separate the truncated subsystem

$$\dot{p}_0 = \mu\beta_0(\varphi_0), \quad \dot{\varphi}_0 = \mu p_0 \tag{1.12}$$

with the same initial conditions.

If the coefficients of the initial system (1.1) can be represented in the form of trigonometric polynomials, the coefficient $\beta_0(\varphi)$ can be regarded as a 2π-periodic zero-mean function [1, 11]. Here the unperturbed system (1.12) has a first integral, defined by the relation ($p = p_0$, $\varphi = \varphi_0$)

$$H = p^2/2 + U(\varphi), \quad dU/d\varphi = -\beta_0(\varphi) \tag{1.13}$$

A typical 2π-periodic potential $U(\varphi)$, characterizing the motion near resonance, is shown in Fig. 1, and the phase trajectories corresponding to it are shown in Fig. 2 [11]. In the phase plane we distinguish domains of oscillatory motion (O) and rotational motion (R), separated by the separatrice S. In other words, the motion of the perturbed system can be regarded as the oscillatory and rotational motions of an equivalent pendulum with a restoring moment $-\beta_0(\varphi)$ and a perturbing torque. Only bounded motions in the domain of the oscillatory motions correspond to "capture in resonance". In terms of the change in total energy H, motions within the potential well W: $\min U(\varphi) < H < \max U(\varphi)$ correspond to the "trapped" motions.

We will investigate the effect of random perturbations on these motions.

Following [12], we will introduce the new variables H and ϕ, where H satisfies (1.13), while the phase ϕ is given by the relations

$$\frac{\partial\phi}{\partial\varphi} = \frac{2\pi}{T(h)}\frac{1}{p(h,\varphi)}, \quad T(H) = \int_{\Gamma(h)} \frac{1}{p(h,\varphi)}d\varphi \tag{1.14}$$

Here $T(H)$ is the period and $2\pi/T(H) = \omega(H)$ is the natural frequency of the oscillations. In the domain 0 the integration is carried out over the closed contour $\Gamma(H)$, bounded by the turning points $\varphi(H)$, and $\bar{\varphi}(H)$ are the roots of the equation $H = U(\varphi)$.

By considering H as a new slow variable, we obtain

$$p(H,\varphi) = \pm\{2[H - U(\varphi)]\}^{1/2} \tag{1.15}$$

Using Ito's formula for the change of variables in stochastic systems [9], we obtain

$$\begin{aligned}\dot{H} &= \frac{\partial h}{\partial\varphi}\dot{\varphi} + \frac{\partial h}{\partial p}\dot{p} + \frac{1}{2}\mu^2\sigma_0^2(\varphi)\frac{\partial^2 h}{\partial p^2} \\ \dot{\phi} &= \frac{\partial\phi}{\partial\varphi}\dot{\varphi} + \frac{\partial\phi}{\partial h}H + \frac{1}{2}\mu^2\sigma_0^2(\varphi)p^2(H,\varphi)\frac{\partial^2\phi}{\partial h^2}\end{aligned} \tag{1.16}$$

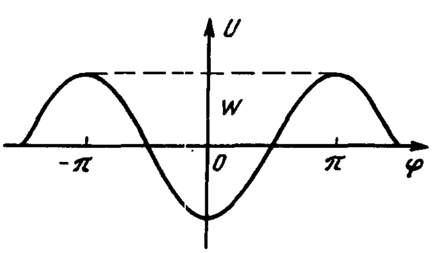

Fig. 1.

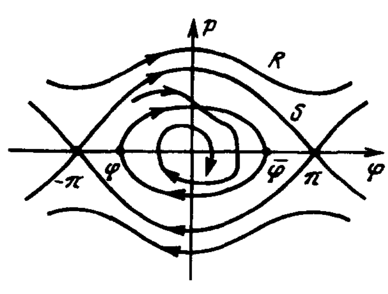

Fig. 2.

It follows from (1.11) and (1.16) that

$$\dot{H} = \mu^2 d(H, \varphi(H, \phi)) + \mu\delta(H, \varphi(H, \phi))\dot{w}(t) \tag{1.17}$$

$$\dot{\phi} = \mu\omega(H) + \mu^2\Psi(H, \varphi(H, \phi)) + \mu\Delta(H, \varphi(H, \phi))\dot{w}(t)$$

where

$$d = (H, \varphi) = \beta_0(\varphi)p^2(H, \varphi) - \beta_1(\varphi)\varkappa_0(\varphi) + \frac{1}{2}\sigma_0^2(\varphi), \quad \delta(H, \varphi) = \sigma_0(\varphi)p(H, \varphi) \tag{1.18}$$

The coefficients Ψ and Δ can be calculated by (1.16). It is obvious that ϕ can be regarded as a fast phase with respect to the slow variable H and, by (14), the right-hand sides of (1.17) are 2π-periodic in ϕ. The averaging principle [13] holds for systems of type (1.17): as $\mu \to 0$ the process $h(t, \mu) \in W$ converges weakly to the slow diffusion process $h(\tau)$, which satisfies the equation

$$h'(\tau) = d_0(h) + \delta_0(h)w'(\tau), \quad \tau = \mu^2 t \tag{1.19}$$

where the prime denotes differentiation with respect to the slow variable τ, and the drift and diffusion coefficients are calculated by averaging over the fast phase ϕ. Taking relation (1.14) between the phases φ and ϕ into account, we obtain

$$d_0(h) = \frac{1}{T(h)} \int_{\Gamma(h)} \frac{d(h, \varphi)}{p(h, \varphi)} d\varphi, \quad \delta_0^2(h) = \frac{1}{T(h)} \int_{\Gamma(h)} \frac{\delta^2(h, \varphi)}{p(h, \varphi)} d\varphi \tag{1.20}$$

where the integration is carried out over the contour along the corresponding phase trajectory (compare (1.14)). A similar averaging scheme is described in [3, 13].

Convergence is ensured in the time interval $\tau \in [0, T] = I_T$ such that $h(\tau) \in W$, $\tau \in I_T$ [13].

2. THE NECESSARY CONDITION FOR TRAPPING

We will formulate conditions similar to the "capture in resonance" conditions (in the probability sense). It is obvious that an effect similar to "capture" occurs if any trajectory, beginning in the region R, falls in the domain O with unit probability in a finite time and, beginning at a certain finite instant of time, does not leave this domain (Fig. 2). This implies [9] that the process $P(t, \mu)$ is positively recurrent with respect to the domain O and non-recurrent with respect to the domain R uniformly for sufficiently small μ. Recurrence problems for perturbed diffusion Hamiltonian systems were discussed previously in [2, 3], but the conditions obtained can only be checked easily for quasi-linear systems with one degree of freedom and time-independent diffusion coefficients. In this problem it is convenient to consider other conditions which, when satisfied, enable the system to leave the domain O.

If we consider the energy $h(\tau)$ as a measure of the deviation from the resonance surface, the condition for emergence reduces to the obvious inequality

$$d_0(h) > 0, \quad h \in W, \quad h(0) \in W \tag{2.1}$$

It follows from (1.18) and (1.20) that (2.1) is weaker than its deterministic counterpart. This implies that random perturbations can cause the system to leave resonance even when the deterministic system has a stable resonance mode. An example is discussed in Section 4.

3. MULTIFREQUENCY SYSTEMS

We will consider some details of the analysis of near-resonant motions in multifrequency systems. We will investigate a system of the form

$$\dot{x} = \varepsilon f(x, \theta) + \varepsilon F(x, \theta)\xi(t), \quad x \in R_n \tag{3.1}$$

$$\dot{\theta} = \omega(x) + \varepsilon g(x, \theta) + \varepsilon G(x, \theta)\xi(t), \quad \theta \in R_m$$

Here $\xi(t)$ is a vector random process which satisfies the conditions of Section 1, f, g and ω are vectors, and F and G are matrices of appropriate dimensionalities. The right-hand sides of (3.1) are assumed to be fairly smooth with respect to their variables and 2π-periodic with respect to each of the components of the vector θ. The non-resonant case was considered in [5].

We will assume that, in the unperturbed system, there is at least one resonance surface

$$\gamma(x) = (\lambda, \omega(x)) = 0 \tag{3.2}$$

where λ is an integral vector, $|\lambda| \neq 0$. If it is possible for several resonances to occur in the approximation considered, we will assume that, for sufficiently small ε, there is no overlap of the resonances, and the motion in the $\sqrt{\varepsilon}$-neighbourhood of each of the surfaces can be investigated independently [11].

As in Section 1, we will introduce new variables, characterizing the motion in the μ-neighbourhood of (3.2), $\mu = \sqrt{\varepsilon}$

$$\mu P = (\lambda, \omega(x)), \quad \Phi = (\lambda, \theta), \quad Y_i = x_i, \quad \psi_i = \theta_i, \quad i = 1, \ldots, n-1 \tag{3.3}$$

Converting (3.1) using (3.3) and retaining only terms that are important for the further analysis, we obtain a system, similar to (1.9), but with the additional slow variable Y

$$\begin{aligned} \dot{P} &= \mu b_0(Y, \Phi, \psi) + \mu^2 b_1(Y, \Phi, \psi)P + \mu D_0(Y, \Phi, \psi)\xi(t) \\ \dot{\Phi} &= \mu P + \mu^2 k_0(Y, \Phi, \psi), \quad \dot{Y} = \mu^2 r_0(Y, \Phi, \psi) \\ \dot{\psi} &= \Omega_0 + \mu\Omega_1 P + \mu^2 v_0(Y, \Phi, \psi) \end{aligned} \tag{3.4}$$

Averaging system (3.4) over the fast phase ψ we obtain that as $\mu \to 0$ its solution can be approximated (in the weak sense) by the solution of the following system [5]

$$\begin{aligned} \dot{p} &= \mu\beta_0(y, \varphi) + \mu^2\beta_1(y, \varphi)p + \mu\sigma_0(y, \varphi)\dot{w}(t) \\ \dot{\varphi} &= \mu p + \mu^2 \varkappa^0(\varphi), \quad \dot{y} = \mu^2 \rho_0(y, \varphi) \end{aligned} \tag{3.5}$$

We will compare (3.5) with the unperturbed system

$$\dot{p}_0 = \mu\beta_0(y, \varphi_0), \quad \dot{\varphi}_0 = \mu p_0$$

We will assume that $\beta_0(y, \varphi)$ is a 2π-periodic function of φ for each fixed y from the region $y \in D_y$ of the change of variable considered. Then, for each $y \in D_y$ the potential $U(y, \varphi)$ and the phase trajectories corresponding to it have the form shown in Figs 1 and 2.

We will define the new slow variable H by a relation similar to (1.13), with $U = U(y, \varphi)$. By the same transformations as in Section 1, we obtain that the variable $H(t, \mu)$, found from the solutions (3.5), converges weakly to the slow process $h(\tau)$, which satisfies the system of equations

$$h'(\tau) = d_0(h, y) + \delta_0(h, y)w'(\tau), \quad t'(\tau) = \nu_0(h, y) \tag{3.6}$$

Here, as in Section 2, $\tau = \mu^2 t$, the prime denotes differentiation with respect to τ, the coefficients d_0 and δ_0 are defined in the same way as (1.20), and $\nu_0(h, y)$ is obtained by averaging the coefficient $\rho_0(h, \varphi, y)$ over the phase φ for fixed y. Convergence is ensured in the time interval $t \in [0, T/\mu^2]$, and in the same interval the conclusions of Section 2 regarding the possibility of "capture in resonance" hold.

4. EXAMPLE

Consider the resonance mode in the system

$$\ddot{z} + 2z[\dot{z}^2 + (\dot{z}^4 + 4z^2)^{1/2}]^{-1} + \varepsilon^{3/2}\alpha\dot{z} = \varepsilon(\sin\omega t + \xi(t)) \tag{4.1}$$

Here $\xi(t)$ is stationary white noise with spectral density S_0. Following the same approach that was used when considering the deterministic analogue of (4.1) [6], we introduce new variables x, θ_1, θ_2 by the formulae

$$z = x\sin\theta_2, \quad \dot{z} = x^{1/2}\cos\theta_2, \quad \dot{\theta}_1 = \omega$$

which convert (4.1) to the standard form (1.1) and then, following (1.6) and (1.7), the variables

$$\theta_2 - \theta_1 = \Phi, \quad \gamma = x^{-1/2} - \omega = \mu P, \quad \mu = \varepsilon^{1/2}$$

whence we have

$$x^{-1/2} = \omega + \mu P, \quad x^{1/2} = \omega^{-1}(1 - \mu\omega^{-1}P) + \mu^2 \ldots$$

As a result, the initial system is reduced to the form (1.9). On the right-hand side of the first of Eqs (1.9) there is an additional term $\mu^2 l$, which appears due to the dissipative forces of the order of $\varepsilon^{3/2}$ in (4.1). The coefficients (1.9) can be written in the form

$$b_0 = \omega^{-1}R(\Phi+\psi)\cos(\Phi+\psi)\sin\psi, \quad k_0 = \omega^{-1}R(\Phi+\psi)\sin(\Phi+\psi)\sin\psi$$
$$b_1 = -\omega^{-1}b_0, \quad l = -\alpha\omega^{-2}R(\Phi+\psi)\cos^2(\Phi+\psi), \quad D_0 = \omega^{-1}R(\Phi+\psi)\cos(\Phi+\psi)$$

where $R(\psi) = 2(1 + \sin^2\psi)^{-1}$. It is obvious that the averaging procedure on the fast phase ψ remains valid, but on the right-hand side of the first of Eqs (1.10) there is an additional term λ_0, corresponding to the averaging of l over ψ. The approximating system (1.10) takes the form

$$\dot{p} = \mu\beta_0(\varphi) + \mu^2[\beta_1(\varphi)p + \lambda_0] + \mu\sigma_0\dot{w}(t), \quad \dot{\varphi} = \mu p + \mu^2\varkappa_0(\varphi) \tag{4.2}$$

where

$$\beta_0(\varphi) = -2\omega^{-1}r_c\sin\varphi, \quad \beta_1 = \omega^{-1}\beta_0, \quad \varkappa_0 = 2\omega^{-1}r_s\cos\varphi$$
$$\lambda_0 = -2\alpha\omega^{-2}r_c, \quad \sigma_0^2 = 4S_0\omega^{-2}\rho^2$$

and (compare [6]) $r_c = \sqrt{2} - 1$, $r_s = 1 - 1/\sqrt{2}$, $\rho^2 = 1/\sqrt{2}$.

We introduce the new variable H by formula (1.13). The potential $U(\varphi) = -2r_c\omega^{-1}$ has the form shown in Fig. 1. Oscillatory modes are defined by the condition $|H| < 2r_c\omega^{-1}$, turning points have the form $\bar{\varphi} = -\varphi = \pi - \arccos(H\omega/(2r_0))$ and the function $p(H, \varphi)$ is defined by (1.15).

Repeating all the transformations in Section 1, we obtain that the approximating diffusion process $h(\tau)$ satisfies the equation

$$h' = d_0 + \sigma_0 V(h)w'(\tau) \tag{4.3}$$

where

$$d_0 = 2\omega^{-2}(2S_0\rho^2 - \alpha r_c) \tag{4.4}$$

$$V^2(h) = \frac{1}{T(h)}\int_{\Gamma(h)} p(h,\varphi)d\varphi, \quad T(h) = \int_{\Gamma(h)} \frac{1}{p(h,\varphi)d\varphi} \tag{4.5}$$

and the integration is carried out over contours in the corresponding regions of variation of the variables. Taking (4.3) into account we obtain

$$V^2(h) = 2\lambda_0^2 E(\bar{\varphi}/2, k)F^{-1}(\bar{\varphi}/2, k) \tag{4.6}$$
$$\lambda_0^2 = h + 2r_c\omega^{-1}, \quad k^2 = 4r_c\omega^{-1}\lambda_0^2$$

where F and E are elliptic integrals of the first and second kind respectively.

By (2.1) and (4.4) the trapping condition $d_0 < 0$ takes the form

$$2S_0\rho^2 < \alpha r_c \tag{4.7}$$

Moreover, inside the region 0 the coefficient $V(h) \neq 0$, i.e. the "trapped" trajectory cannot remain in the equilibrium position $p = 0$, $\varphi = 0$. This means that fluctuations in the frequency of the oscillations $\mu P = x^{-1/2} - \omega$ do not vanish and can be calculated from Eq. (4.3).

Hence, random perturbations contract the trapping region of trajectories, and strict synchronization in the perturbed system becomes impossible.

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